Calculus II - Day 18

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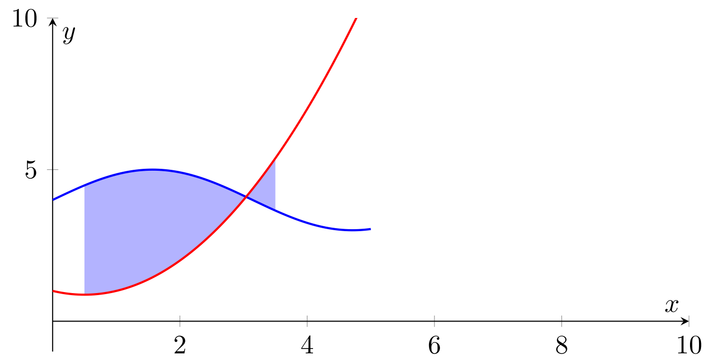
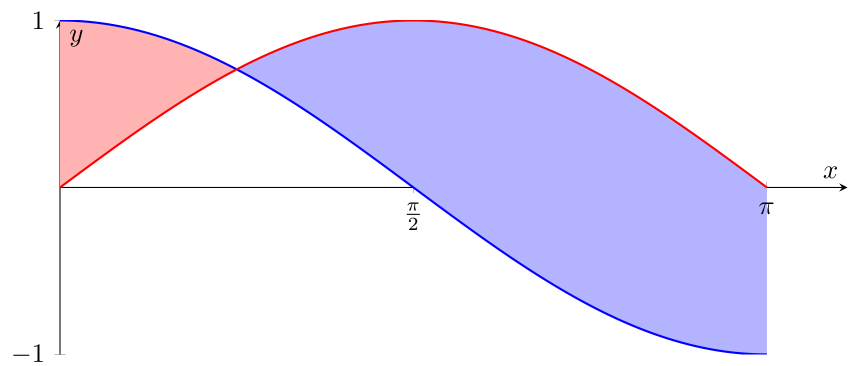
13 November 2024

Goals for today:

- Find the area between curves.
- Compute volumes of 3-D solids by "slicing and summing."
- Compute volumes of solids of revolution using the disk/washer method.

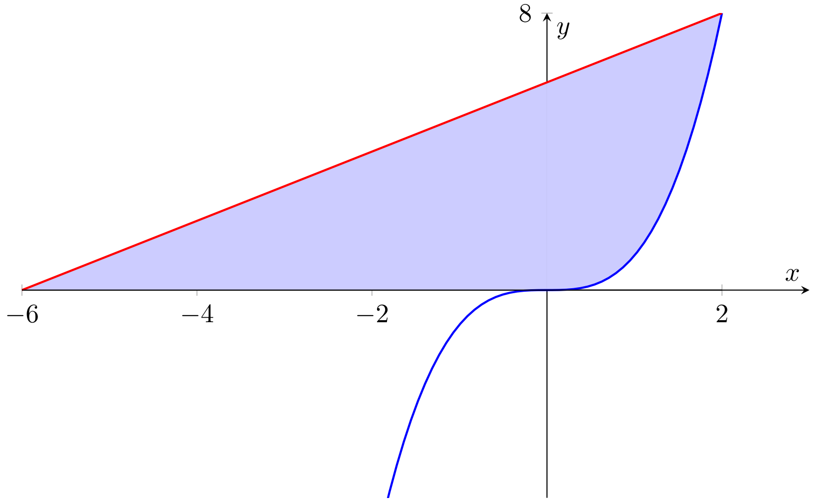


$$\int_a^b (f(x) - g(x))\, dx$$



$$\text{Area} = \int_0^{\pi/4} (\cos(x) - \sin(x))\, dx + \int_{\pi/4}^{\pi} (\sin(x) - \cos(x))\, dx$$
$$= [\sin(x) + \cos(x)]_0^{\pi/4} + [-\cos(x) - \sin(x)]_{\pi/4}^{\pi}$$
$$= (\sqrt{2} - 1) + (1 + \sqrt{2}) = 2\sqrt{2}$$

Ex. Find the area of the region R bounded by the graphs: $y = x^3$, $y = x + 6$, and $y =$



0 (the x-axis).

First approach (integrate with respect to x):

$$\text{Area} = \int_{-6}^{0} (x + 6)\, dx + \int_0^2 \left(x + 6 - x^3\right)\, dx$$

Second approach (integrate with respect to y):

"Top Function":

$$x = y^{1/3}$$

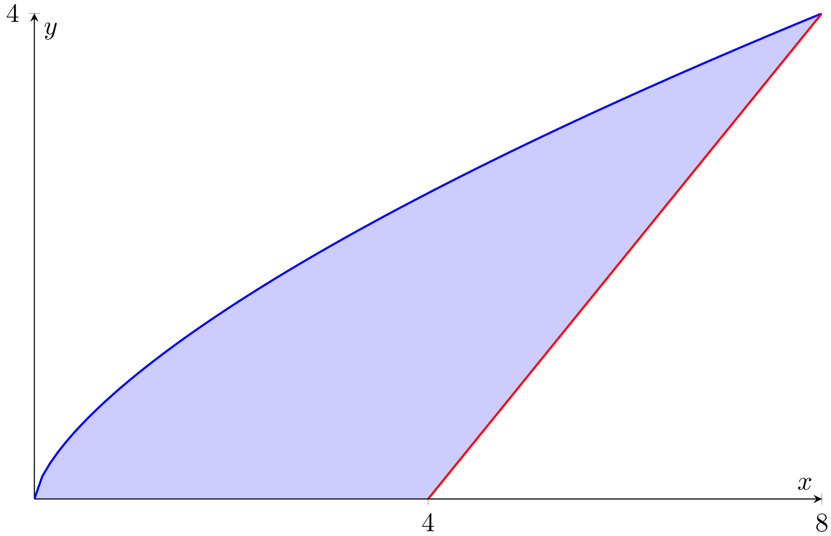
"Bottom Function":

$$x = y - 6$$

on [0, 8]

$$\text{Area} = \int_0^8 \left(y^{1/3} - (y - 6)\right)\, dy$$

Ex. The area between the curves: $y = 0$, $x = 0$, $y = x^{2/3}$, and $y = x - 4$.



First approach (integrate with respect to x):

$$\text{Area} = \int_0^4 \left(x^{2/3} - 0\right)\, dx + \int_4^8 \left(x^{2/3} - (x-4)\right)\, dx$$

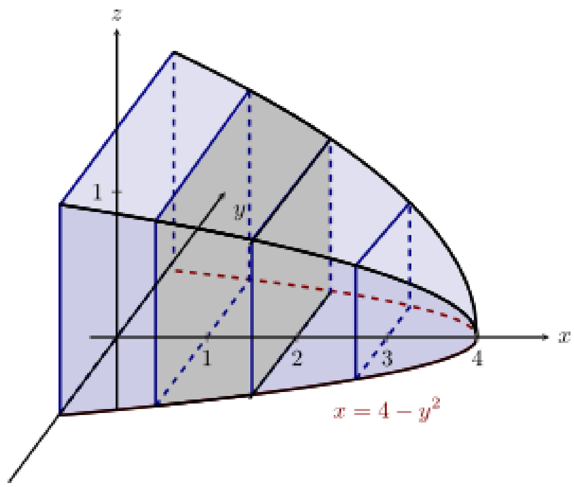
$$= \int_0^4 x^{2/3}\, dx + \int_4^8 \left(x^{2/3} - x + 4\right)\, dx$$

Second approach (integrate with respect to y):

$$\text{Top: } x = y + 4, \quad \text{Bottom: } x = y^{3/2} \quad \text{on } [0, 4]$$

$$\text{Area} = \int_0^4 \left((y+4) - y^{3/2}\right)\, dy$$

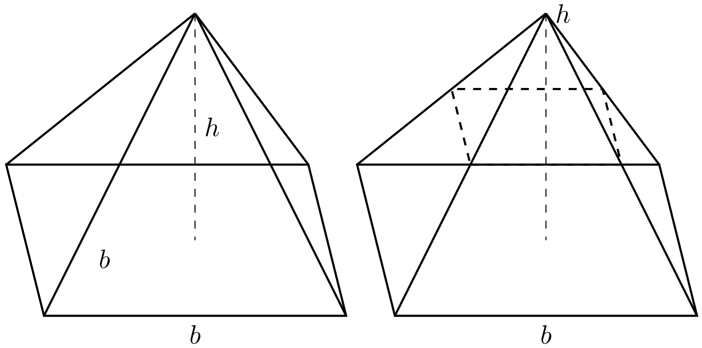
$$= \int_0^4 \left(y + 4 - y^{3/2}\right)\, dy$$



A(x) = area of the vertical cross-section

$$\text{Volume} = \int_a^b A(x)\,dx$$

Ex. Find the volume of a right square-based pyramid with height h and base $b \times b$.



The length of the side increases linearly from $(0, 0)$ to (h, b). Thus, the side length at height x is given by $y = \frac{b}{h}x$. The cross-section area at height x is:

$$A(x) = \left(\frac{b}{h}x\right)^2 = \frac{b^2}{h^2}x^2.$$

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The volume of the pyramid, V, is then:

$$V = \int_0^h A(x)\,dx = \int_0^h \frac{b^2}{h^2}x^2\,dx = \frac{b^2}{h^2}\int_0^h x^2\,dx.$$

Evaluating the integral:

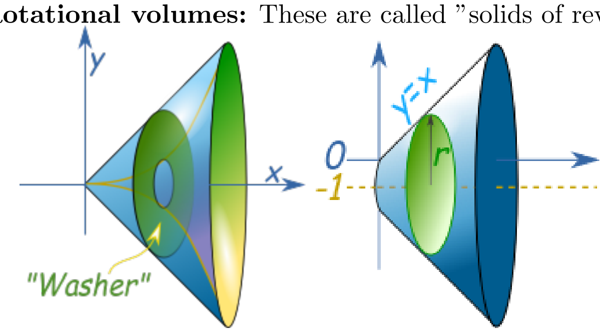
$$V = \frac{b^2}{h^2} \cdot \frac{1}{3} x^3 \Big|_0^h = \frac{b^2}{h^2} \cdot \frac{1}{3} h^3 = \frac{b^2 h}{3}.$$

Volume Calculations

Suppose a solid object extends from $x = a$ to $x = b$ in 3-D space. If the cross-sections of the solid perpendicular to the x-axis have the area $A(x)$, the volume of the solid is

$$V = \int_a^b A(x)\, dx.$$

Rotational volumes: These are called "solids of revolution."



The area of a disk is $A = \pi r^2$. Thus, for a function $f(x)$, the cross-sectional area at x is

$$A(x) = \pi \left(f(x)\right)^2.$$

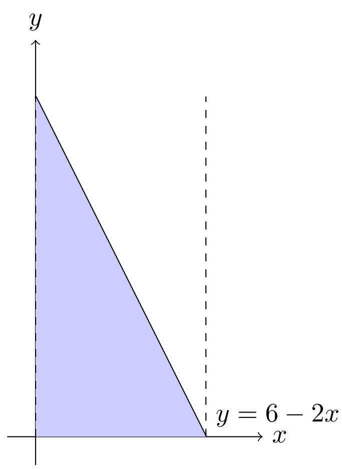
The volume of the solid formed by rotating $f(x)$ about the x-axis from $x = a$ to $x = b$ is

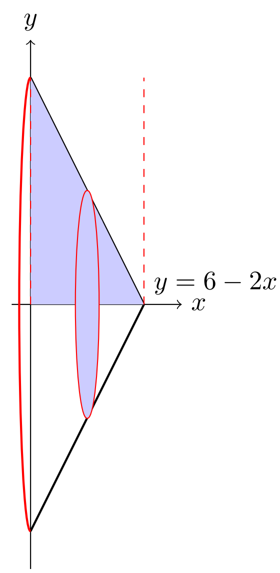
$$V = \int_a^b \pi \left(f(x)\right)^2 \, dx.$$

Let R be the region bounded between the curves $y = 6 - 2x$, $x = 0$, and $y = 0$. Find the volume of the solid of revolution:

(a) about the x-axis

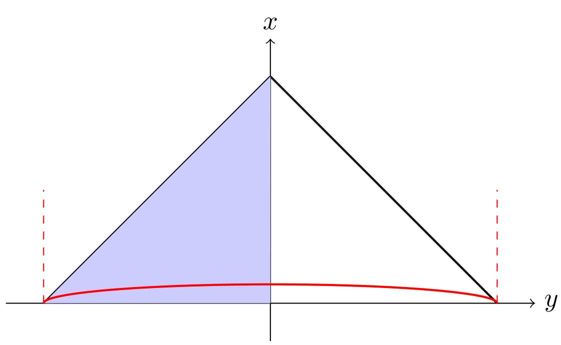
(b) about the y-axis





$$V = \int_0^3 \pi(6-2x)^2\,dx$$
$$= 36\pi$$

Y-Axis



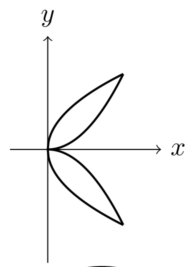
The radius of a disk is determined by $y = 6 - 2x$. Solving for x in terms of y:

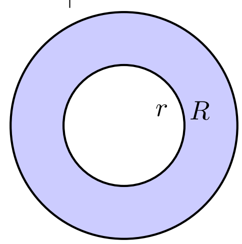
$$2x = 6 - y \Rightarrow x = 3 - \frac{y}{2}.$$

The volume V of the solid formed by rotating the region about the y-axis is:

$$V = \int_{y=0}^{y=6} \pi \left(3 - \frac{y}{2}\right)^2 \, dy = \ldots = 18\pi.$$

What if there's a whole in the middle?

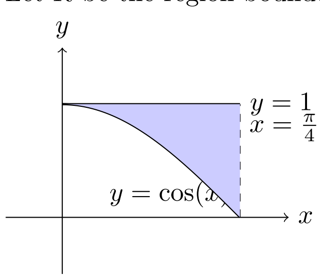




The volume V of a solid of revolution with a central hole, created by rotating the region bounded by $y = x$ and $y = x^4$ about the x-axis from $x = 0$ to $x = 1$, is:

$$V = \int_0^1 \pi \left(x - x^4\right) \, dx$$

Example:
Let R be the region bounded by $y = \cos(x)$, $y = 1$, $x = \frac{\pi}{4}$, and $x = 0$.



The volume of the solid formed by rotating R about the x-axis is given by:

$$V = \int_0^{\pi/4} \pi \left(1^2 - \cos^2(x)\right) \, dx$$

Rotational volume about the y-axis:

$$V = \int_{\frac{1}{\sqrt{2}}}^{1} \pi \left(\left(\frac{\pi}{4} \right)^2 - \arccos(y)^2 \right) \, dy \quad \text{... very hard to integrate!}$$